Math 347: Practice for Final Exam
Dec. 12, 2018

1. Let $n \geq 2$ be a natural number.

   a) Assume n is a prime number. Prove that $x^2 \equiv 1 \mod n$ if and only if $x \equiv 1 \mod n$ or $x \equiv n-1 \mod n$.

   b) Does a) still hold if n is not a prime number? Prove your statement or give a conterexample.

   c) Two siblings were born exactly 15 months apart. Knowing that every year their birthday happens on the same day of the week, find out on which months they could have been born.

2. Let (a_n) be a Cauchy sequence, and (b_n) a subsequence such that $\lim b_n = L$. Prove that $\lim a_n$ exists and is equal to L.

3. a) Let $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$. Prove that there exist n and m such that

$$na + mb = 1.$$

b) Suppose that $\gcd(a, b)$ divides c. Does the equation

$$ax + by = c.$$

have integers solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$?

c) Let $(x_0, y_0) \in \mathbb{Z} \times \mathbb{Z}$ be a solution to

$$ax + by = c.$$

Write all the solutions to the equation above in terms of x_0, y_0, a, b and $d = \gcd(a, b)$.

4. a) Let S be a finite set and $f : S \to S$ an injective function. Prove that f is surjective.

b) Give an example of a set S where the above conclusion fails[1].

c) Let T be a set such that there exists an injective function $g : T \to \mathbb{N}$. What can you say about the cardinality[2] of T.

[1] Namely, where an injective function from S to S is not necessarily surjective.

[2] Be precise in your answer, if necessary recall what we defined about any words you use.

5. Let S be the set of sequences of non-zero real numbers, i.e. functions from $\mathbb{N}$ to $\mathbb{R}\backslash\{0\}$. Consider the relation R on S,

$$((a_n),(b_n)) \in R, \text{ if } \forall \epsilon \in \mathbb{R}_{>0},\ \exists N \in \mathbb{N}, \text{ s.t. } \forall n \geq N,\ |a_n - b_n| < \epsilon.$$

a) Prove that R is an equivalence relation.

b) Give three examples of sequences in the equivalence class of $a_n = \frac{1}{n^2}$.

c) Prove that $a_n = \frac{(-1)^n}{n}$ and $b_n = \frac{1}{n}$ are in the same equivalence class.

6. a) For $n \geq 1$, prove that[3]

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

b) Find the number of non-negative integer solutions to

$$x_1 + x_2 + \cdots + x_k \leq n.$$

[3] If $a < b$ we define

$$\binom{a}{b} = 0.$$

7. Give examples of the following structures or argue why no example exists. You also need to explain why your examples satisfy the required properties.

   a) A set S and a relation R, that is symmetric and reflexive but not transitive.

   b) A set S and a relation R, that is reflexive, transitive and anti-symmetric, i.e. if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

   c) An equivalence relation R on $\mathbb{Z}$ that has finitely many equivalence classes and an equivalence relation R' that has infinitely many equivalence classes.

   d) Two functions f and g, such that $g \circ f$ is surjective but f is not surjective.

e) A non-monotone Cauchy sequence.

f) Sets A, B, a function $f : A \to B$, and a subset $T \subseteq B$, such that

$$f(f^{-1}(T)) \neq T.$$

g) Sets A, B and C such that

$$(A \cap B) \cup C = A \cap (B \cup C).$$

h) Sets S such that the set of functions $X \to S$ has the same cardinality as $P(X)$, the power set of X.

8. Determine if the following are true or false, and give a brief explanation.

a) The statements

$$(P \Rightarrow (Q \wedge \neg Q)) \Rightarrow \neg P$$

and

$$P \vee \neg P$$

are logically equivalent.

b) Fix $a, L \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$. Consider the statements

$$P = (\forall \epsilon)\ (\exists \delta > 0)\ (\forall x \in \mathbb{R})\ [(0 < |x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon),$$

and

$$Q = (\exists \delta > 0)\ (\forall \epsilon)\ (\forall x \in \mathbb{R})\ [(0 < |x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

Then $Q \Rightarrow P$.

c) For any $n \in \mathbb{Z}$ and any $b \in \mathbb{Z}\backslash\{0\}$, there exists an unique pair $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$n = bq + r.$$

d) Let $n, a \in \mathbb{N}$. Consider the statements

$$P = (\exists x \in \mathbb{N})(a + x \equiv 1 \bmod n),$$

and

$$Q = ([a]+ : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \text{ is surjective.})^{4}.$$

Then $Q \Rightarrow P$ is false.

e) The statement

$$(\forall n \in \mathbb{N})(\gcd(n, n+3) = 1)$$

is true.

f) Let $A \subset B$ be two finite sets. Suppose that there are 4 subsets of B containing A. Then $|A| = |B| - 2$.

[4] Recall that this function is explicitly defined as $[a] + [x] = [a + x]$, for any $[x] \in \mathbb{Z}/n\mathbb{Z}$.